Knots and Links v3.0

Structures and Classifications

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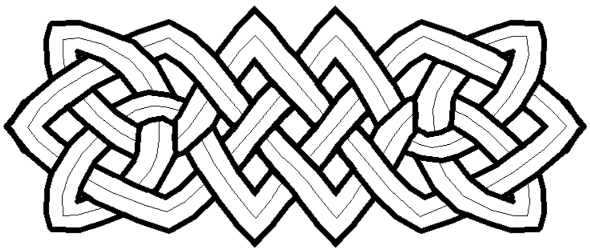
Topsail Schooner "Albatros", DEQZ 2015

Abstract

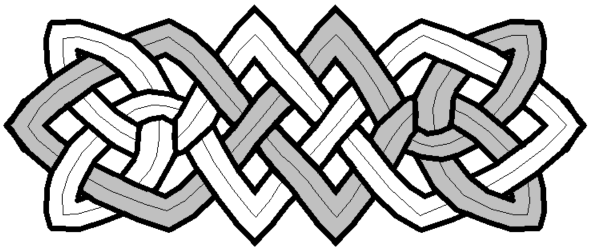
This document deals with mathematical knots and links. It standardizes knot graphs instead of utilizing the usual knot diagrams. Operations are defined to recognize equivalent knot graphs. Furthermore, operations are defined to generate knot graphs with higher number of crossings from graphs with lower number of crossings. Methods are described to reduce knot graphs to graphs with lower number of crossings to distinguish between graphs of knots and links. These methods and operations establish order relations and equivalence relations on the set of knot graphs. It turns out that the set of knot graphs is divided into two disjunct subsets of graphs representing knots or links, respectively, but the order structure includes both subsets. One equivalence relation parts the set of graphs representing knots, dividing them into classes of prime knots. The reduction method creates another equivalence relation on the whole set of knot graphs which represent knots. It seems that this relation has only two classes: One is represented by the trifoil knot the other by the figure eight knot. This document includes tables of knot and link graphs up to seven crossings and some graphs with higher crossing numbers. This work imposes struktures and classifications on the set of knot graphs but addresses also further problems. The main problem is the use of ambivalent knot graphs instead of knot diagrams. Therefore a lot of open questions remain.

A final remark: This work is pure mathematics and does not search for new utility knots or practical applications - even if the author as a sailor and mariner would be highly interested in such matters.

Knot or Link?



Link!



Contents

1 Introduction

Knots belong to the craft of every sailor. But the goal of this document is pure mathematical: How can knots be distinguished from links? Does the set of knots show mathematical structures or can knots be classified? To approach these questions standardized knot graphs and a related formal terminology and nomenclature are defined instead of the usual knot diagrams. Classical graph theory is of little help because e.g. in a prime knot no crossing is connected to itself, therefore the adjacent matrix of the related knot graph has the main diagonal occupied with zeroes and the number of edges connected with a node is always four.

In this document operations are desribed that allow to recognize equivalent knot graphs. Furthermore, operations are defined to generate knot graphs with higher number of crossings from graphs with lower number of crossings. A reduction method is introduced which allows to distinguish between knot graphs and link graphs. It turns out that these operations and methods constitute an order relation and equivalence relations on the set of knot graphs that allow to structurize the set of knot graphs. Some theorems and conjectures are formulated. But the main problem is the use of ambivalent knot graphs instead of knot diagrams. Therefore a lot of open questions remain.

Because some of the image headlines contain not translated German text, here a glossary for the easier understanding is added.

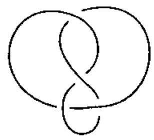
2 Glossary

German	English
Achtknoten	Figure Eight Knot
Anzahl	Number
aktuell	actual
aus	from
Austörnen	Turning, twisting
durch	by
Erweiterung	Extension, Expansion
Klappung	Wrapping
Kleeblattknoten	Trefoil Knot
Knoten	Knot
Kreuzung	Crossing, intersection
längs	along
nichtprim	non-prime
Nullknoten	Unknot
Palstek	Bowline Hitch
Potenz	Power
prim	prime
quer	across
Quotient	Quotient
Reduktion	Reduction
reduzierbar	reducable
Rotation	Rotation
Schotstek	Sheet Bend
Streckung	Streching
Teilung	Splitting
Törnen	Turning, twisting
Trendlinie	Trend line
Türkischer Bund	Turks Head Knot
Überhandknoten	Overhand Knot
und	and
Ursprung	Origin
Verdrehen	Twisting
Verhältnis	Relation
Verschiebung	Shift
Verschlingung	Link
Webeleinstek	Clove Hitch
Wuling	Tangle
Zweieck	Bi-angle

3 Knot Diagrams and Knot Graphs

Knot graphs (also called "Knot Shadows") are used instead of the usual, flat knot diagrams:

The Figure-Eight-knot as a knot diagram:



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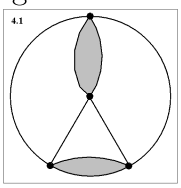
Because all knot diagrams represent closed space curves, knot graphs can always be drawn in a circle. Crossings are represented by points:

The Figure-Eight-knot as a knot graph:



There are ambiguities of knot graphs, which are secondary by now, because we deal primarily with the distinction of knots and links by means of knot graphs and also the recognition of the same knot in different knot graphs.

Bowline Hitch and Sheet Bend variants ...

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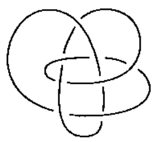
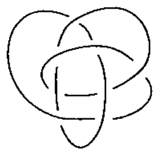


8_{19}



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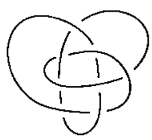

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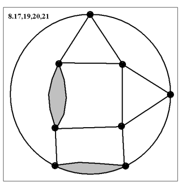
... are projected onto one single knot graph:

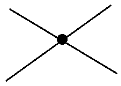


3.1 Definitions

The following expressions are used to designate the parts of the knot graph:

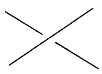
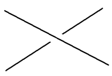
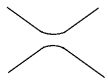
Definition: **Strand** instead of the graph-theoretical "edge":

Definition: **Crossing** or **Intersection** instead of the graph-theoretical "vertex":



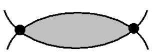
Exactly four strands depart from every intersection (crossing). However, no crossing is connected by a strand to itself, since by simple turning the strand such a crossing can be deleted.

Due to the ambiguity of the crossings in the knot graph even so knot graphs are ambiguous. A crossing in the knot graph can be shown in two ways:

So  or so  but never so 

Each knot graph can be transformed into a knot diagram, if the strands are followed in one direction and at each intersection alternating leading over or under the opposite strand. For some knots, there are however no alternating solutions, e.g. K8.19 .

Definition: A **Bi-Angle** arises if two crossings have two joint strands:

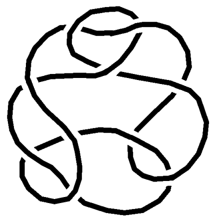


Definition: A **tangle** is neither a knot nor a link, but can be disentangled into the unknot. It is so to speak a two-dimensional knotting of the trivial or unknot, whereas a real knot is always knotted in three dimensions.

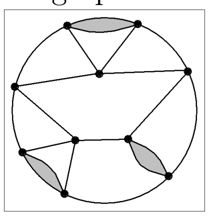
The corresponding knot graph can however be drawn into the diagram of a real knot with higher number of crossing because of its ambiguity.

Example:

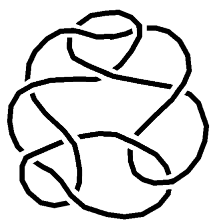
Tangle:



The knot graph of a tangle:



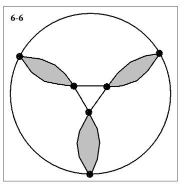
Knot made from the knot graph of a tangle:



3.2 Observations

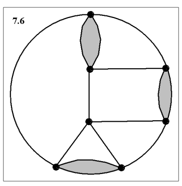
- Knot graphs have the advantage that structures and similarities are easier to discover than in the somewhat chaotic knot diagrams.
- Knot graphs are planar graphs. That means that no strand is crossing another strand. Strands only connect crossing points.
- Different knots may be projected onto the same knot graphs as there the directions of the crossings are not included (example: knot 8.17, 8.19). This is however not decisive for the problem since it is only to distinguish knots from links.
- A knot graph has only crossings of the even degree 4 because from any crossing exactly four strands depart (strictly speaking a crossing is formed by two strands going on top of each other).
- There is only one trail of crossing strands in the knot graph if a knot is represented.
- Even if the knot graph looks coherent and its crossings have even degree it can be closed unicursal or closed multicursal. In the second case, the knot graph depcits a link.
- **Definition:** Degree of a link: number of closed loops that form the link.

Link of degree 3:



- A knot is so to speak a link of the 1st degree.

Knot no. 7.6:

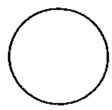


4 Operations

Here is described how by three extension operations from a given knot graph new knot graphs can be generated with an increased number of crossings. This can be both, graphs of knots as well as graphs of links. Furthermore, by the means of equivalence operations can be detected which graphs represent the same knot or same link.

With the operations described in this chapter it is not yet possible to distinguish between knots and links.

Starting point of all extensions is the **Unknot** a knot without intersections:



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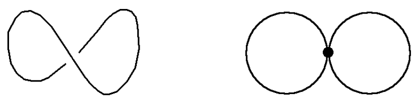
A (reducable) knots with one intersection is created by **Turning** the strand of the unknot:



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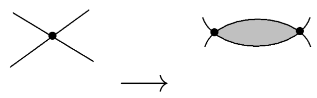
This knot can be produced also by means of the splitting rule (see below chap. 4.1.3).

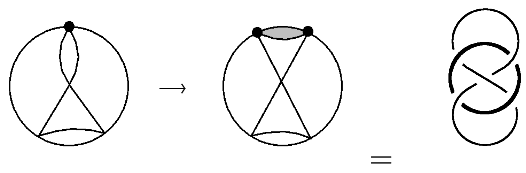
4.1 Extensions

To create new knot graphs with higher number of crossings, a crossing can be **expanded** in two ways into bi-angles: A crossing can be split into a bi-angle either "along" or "across". The third way of expansion is to insert new crossings into strands. The generated knot graph may represent a knot or a link, regardless of whether the predecessor was a knot or a link.

4.1.1 "Along"

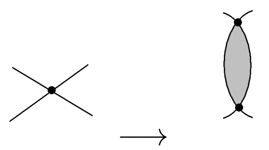


Example:

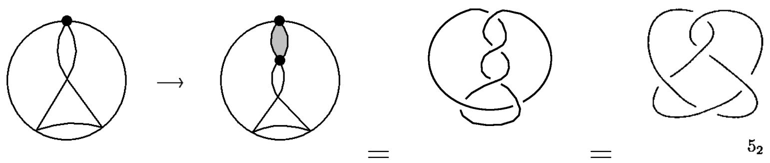


Hereby the figure eight knot becomes a link of second degree.

4.1.2 "Across"



Example:



This expands the figure eight knot K4.1 into the knot K5.2 of the conventional knot tables.

4.1.3 "Splitting"

The strands of a knot graph can be "split" according to following rules with new crossing points and the knot graph then be extended:

1. "Split" of a strand by inserting a new crossing point.

2. Each existing strand can be splitted only once. It can be inserted exactly one crossing per strand.

3. The new crossings have to be connected by new strands so that...

3.1. ... all crossings are connected by exactly four strands and...

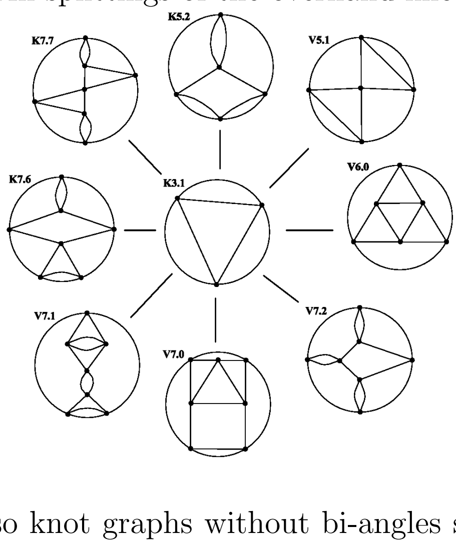
3.2. ... no new strand crosses existing strands (which would violate rule 2).

4. So many as desired of the maximum possible new crossings can be inserted.

5. Only non-reducable knots may be split, except for the knot K1.1.!

Example:

All splittings of the overhand knot



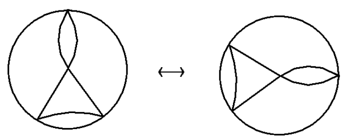
By these rules also knot graphs without bi-angles such as e.g. V6.0 can be generated.

4.2 Equivalence Operations

Equivalence operations leave a knot unchanged but transform the knot graph into another equivalent one. In particular, it retains the number of crossings. These operations are based on the possibilities how real knots can be deformed (see chapter 8.7).

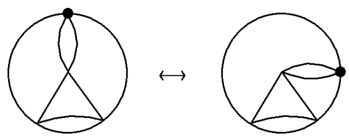
4.2.1 Rotation

Rotate the entire knot graph about a fixed but arbitrary angle, for example:



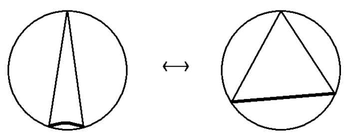
4.2.2 Shift

Shift crossing points along strands, for example:



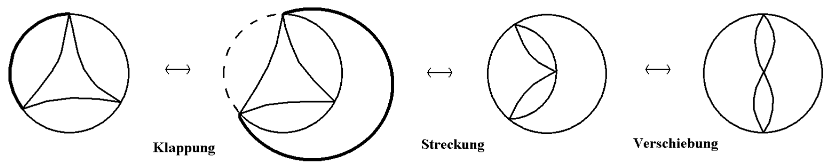
4.2.3 Scaling

Scaling (streching or shortening) of strands, e.g.:



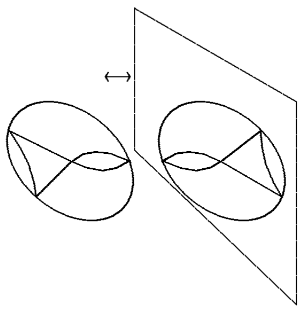
4.2.4 Wrapping

Wrap a strand at the edge of the knot graph abround the entire graph, for example:



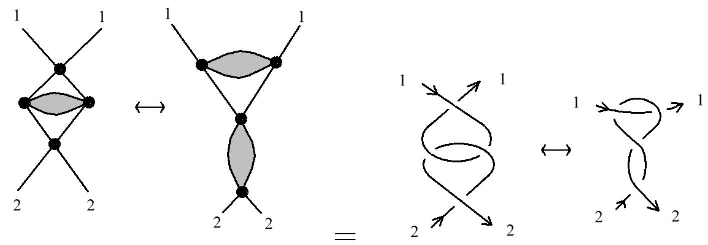
4.2.5 Mirroring

Mirror the entire knot graph, for example:



4.2.6 Twisting

Twist strands of the knot (and therefore its knot graph) while retaining the crossing number and the running directions of the strands:



The most important equivalence operations are wrapping and twisting.

5 Detecting Links

There are two methods for detecting links in knot graphs:

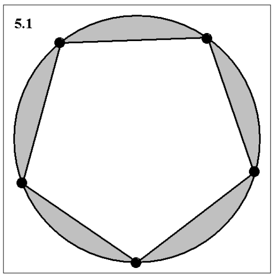
1. Special patterns that serve as criteria for links.
2. Detection of the features mentioned in 1) after reducing the number of crossings without changing the running direction of the strands. I.e. simplification of the knot graph without surrender of essential properties while it is reduced to known graphs of knots or links.

All knot graph which represent no links are knots, except for equivalence.

5.1 Characteristics of Links

The first topological pattern is very easy to detect, if the graph is a closed chain of bi-angles:

If the number of bi-angles in the closed chain is odd it represents a knot:



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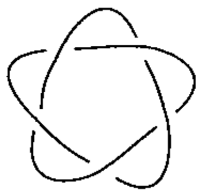

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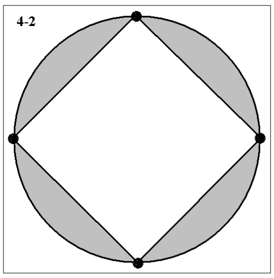
otherwise it represents a link:



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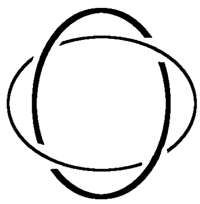
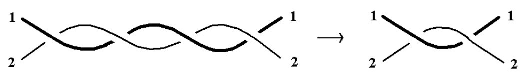


5.2 Reduction Method

In the following sequences the strands have the same direction but fewer intersections. Therefore, they are suitable for the reduction of knot graphs.

5.2.1 Reduce Three Bi-Angles to One

A chain of three bi-angles can be reduced to a bi-angle, four crossings are reduced to two:



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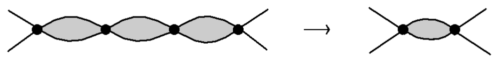
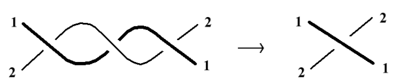


5.2.2 Reduce Two Bi-Angles to One Crossing

Two bi-angles can be reduced into one crossing, three crossings are reduced to one:



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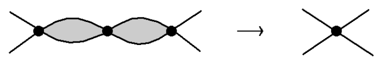
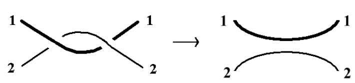


5.2.3 Resolution of Bi-Angles

A bi-angle can be dissolved to vanish:



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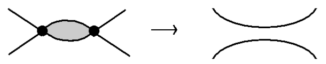
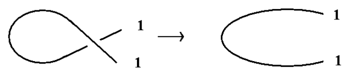


5.2.4 Twisting of a Bight

A bight (a crossing connected by one strand to itself) can be twisted to vanish:



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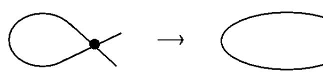
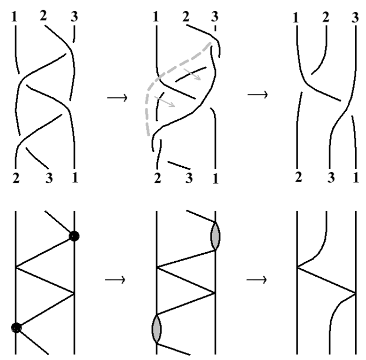


5.2.5 Reduction after Expansion

For knot graphs with no bi-angles, like for example the Turk Head Knot, its own reduction rule is needed: By wrapping of a strand, temporarily more crossings are generated, but also bi-angles are produced, which then can be reduced.

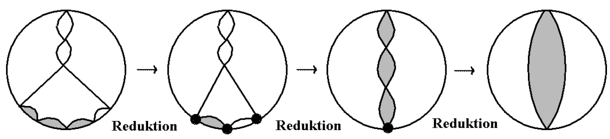


If knot graphs are reduced with the above rules to a smaller number of crossing, possibly using equivalence operations, then it may be possible to recognize topological patterns or the degree of links, or the reduction is progressing until an elementary knot with lower crossing number is recognized.

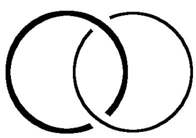
Because the direction and coherence of the strands in the knot are not changed by the reduction, the topological properties of the finally reduced knot can be transferred to the more complex knot.

5.2.6 Examples

Examples to illustrate the method:

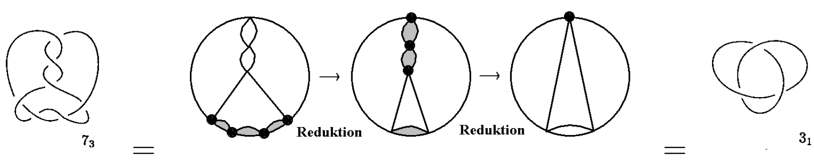


Die Reduktion führt auf den Graphen der einfachsten Verschlingung

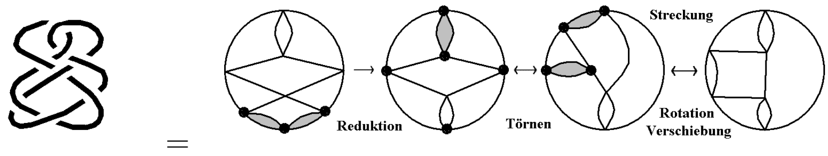


The reduction leads to the graph of the simplest link and thus also the original graph represents a link.

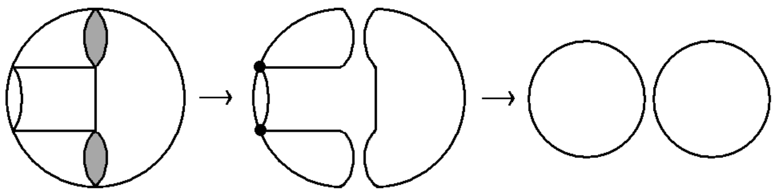
The following reduction leads to the overhand knot, so that here the original graph represents a knot:



The following example shows a more complex reduction with twisting and shifting:

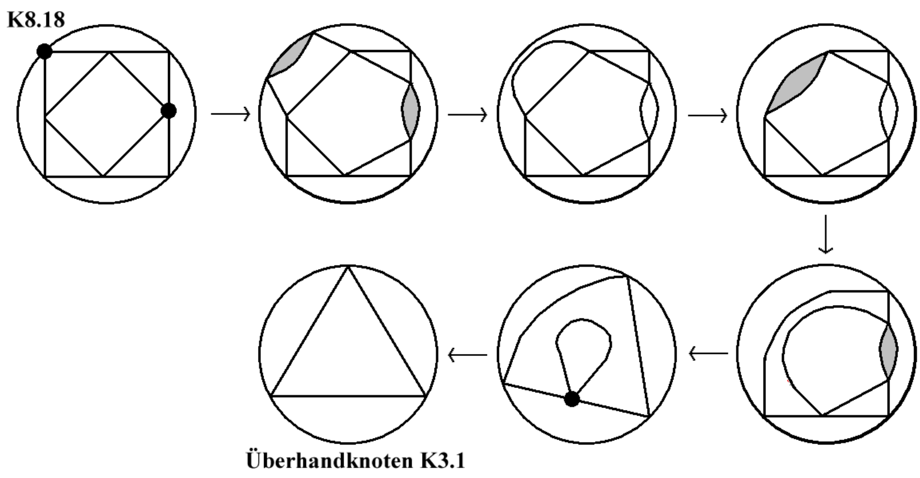


In the last graph the topological pattern of a link is recognizeable in the right half:

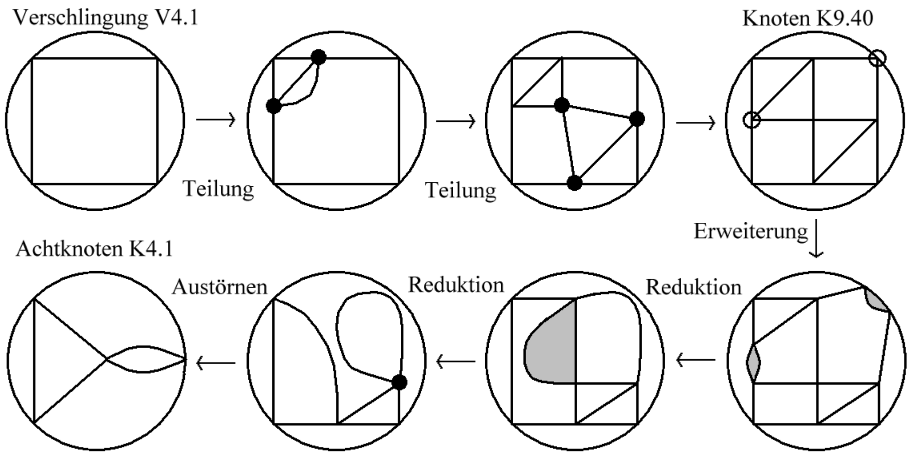


The reduction leads to a link of 2nd degree. Therefore, the original graph is also a link of 2nd degree.

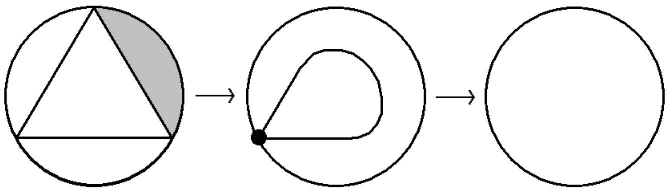
The Turk Head Knot (knot K8. 18) is reduced to the overhand knot:



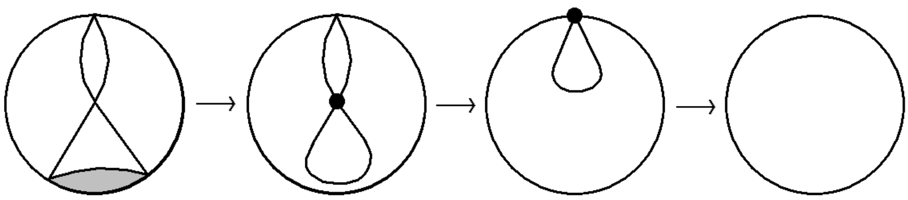
And as another example of the reduction by extension it is shown that the knot K9. 40 reduces to the figure eight knot:



The overhand knot is reduced directly to the unknot:



The figure eight knot reduces also directly on the unknot:



6 Try of a Nomenclature

The model of this nomenclature for knot graphs is the nomenclature, used for the description of complex organic chemical compounds.

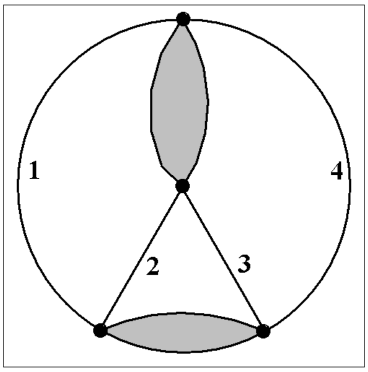
Because knot graphs consist of much fewer items, compared to chemical elements (only three: crossings, strands, bi-angles) and the number of connections is manageable (exactly four strands leave each crossing), therefore it should easily be possible to find simple, structure-descriptive nomenclature rules which can be translated into a knot graph and vice versa.

The rules of the nomenclature should firstly describe the structure of the knot graph, and it should easily be possible formally to perform transformations and operations. If then the structural formula even allows to identify topological patterns, the set of rules is perfect. Possibly algebraic structures (groups, rings, etc.) or other properties can be identified and derived.

6.1 Elements

If a knot graph should be translated into a structural formula, first the strands are numbered randomly. Crossings are denoted by K and bi-angles with Z.

Knot graph with numbered strands:



This knot graph contains two bi-angles where two strands come from each of their two corners. The description of one of the bi-angles looks like this:

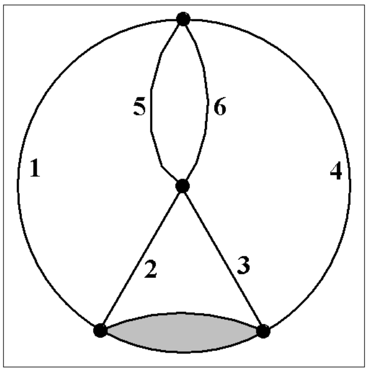
1 2 Z 3 4

and of the other:

1 4 Z 2 3

But a bi-angle can also be represented by two crossings, connected by two strands:

Alternative representation of a bi-angle:



The knot graph now contains two crossings K, one of which together with its strands is described by

1 5 K 6 4
the other by

2 3 K 5 6

Because the numbering of the strands is arbitrary and no orientation in crossings is given, there are several variants how a crossing can be described:
a b K c d
b a K c d
a b K d c
b a K d c
a c K b d
c a K b d
a c K d b
etc.

The description of the knot graph is an arbitrary sequence of the descriptions of the crossings and bi-angles and their strands and therefore can be written as follows:

1 5 K 4 6 2 3 K 5 6 1 2 Z 3 4 = 1 4 Z 2 3 1 2 Z 3 4

The description can easily and uniquely (except for equivalence) be retranslated in a knot graph.

6.2 Operations

The extension of an intersection into an "along" bi-angle looks as follows in the structural description:

a b K c d $\rightarrow$ a b Z c d.

The extension of an "across" crossing into a bi-angle has the following structural description:

a b K c d $\rightarrow$ a c Z b d.

With bi-angles, there are only the following description variants

a b Z...

b a Z...

and at the other end similar.

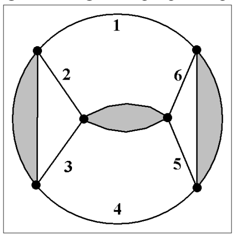
bi-angle chains are represented by ZZ... .

There are numerous descriptions of knot graph, however, all uniquely describe the structure.

6.3 Examples

Original link:

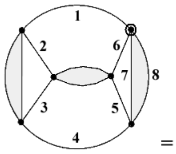
1 2 Z 3 4 2 3 Z 5 6 4 5 Z 1 6



This link consists of

1. a bi-angle to which connect on one side the strands 1 and 2 and to the other the strands and 3 and 4 (1 2 Z 3 4),
2. a bi-angle to which connect on one side the strands 2 and 3 and to the other the strands and 5 and 6 (2 3 Z 5 6) and
3. a bi-angle to which connect on one side the strands 4 and 5, and to the other the strands 1 and 6 (4 5 Z 1 6).

Erweiterung quer

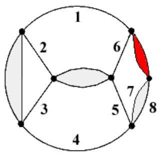


1 2 Z 3 4   2 3 Z 5 6   4 5 Z 1 6

= 1 2 Z 3 4   2 3 Z 5 6   4 5 K 7 8   7 8 K 1 6

↓

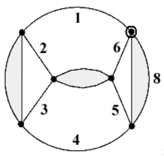
1 2 Z 3 4   2 3 Z 5 6   4 5 K 7 8   7 8 Z 1 6



1 2 Z 3 4   2 3 Z 5 6   4 5 Z Z 1 6

7_5

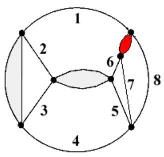
Erweiterung längs



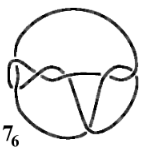
1 2 Z 3 4   2 3 Z 5 6   4 5 Z 1 6

= 1 2 Z 3 4   2 3 Z 5 6   4 5 K 7 8   7 8 K 1 6

↓



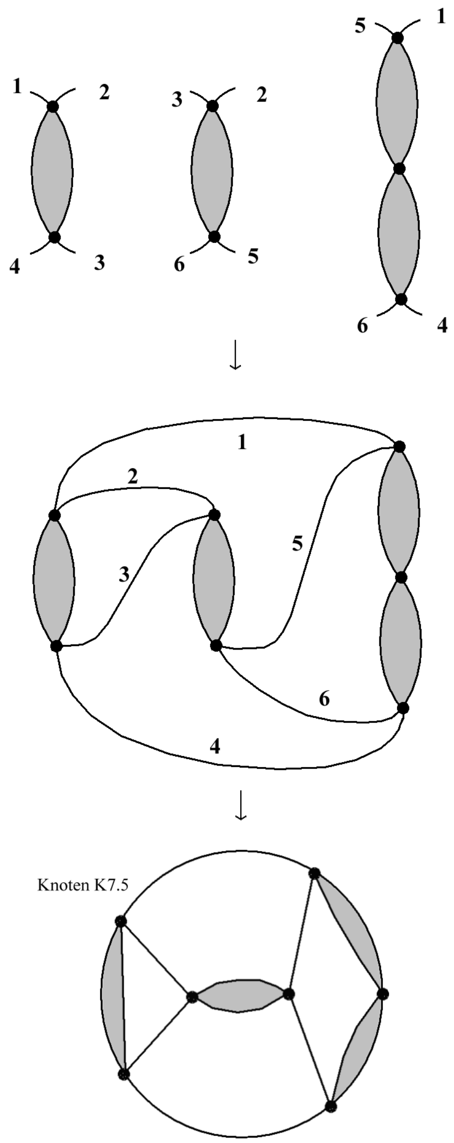
1 2 Z 3 4   2 3 Z 5 6   4 5 K 7 8   6 7 Z 1 8

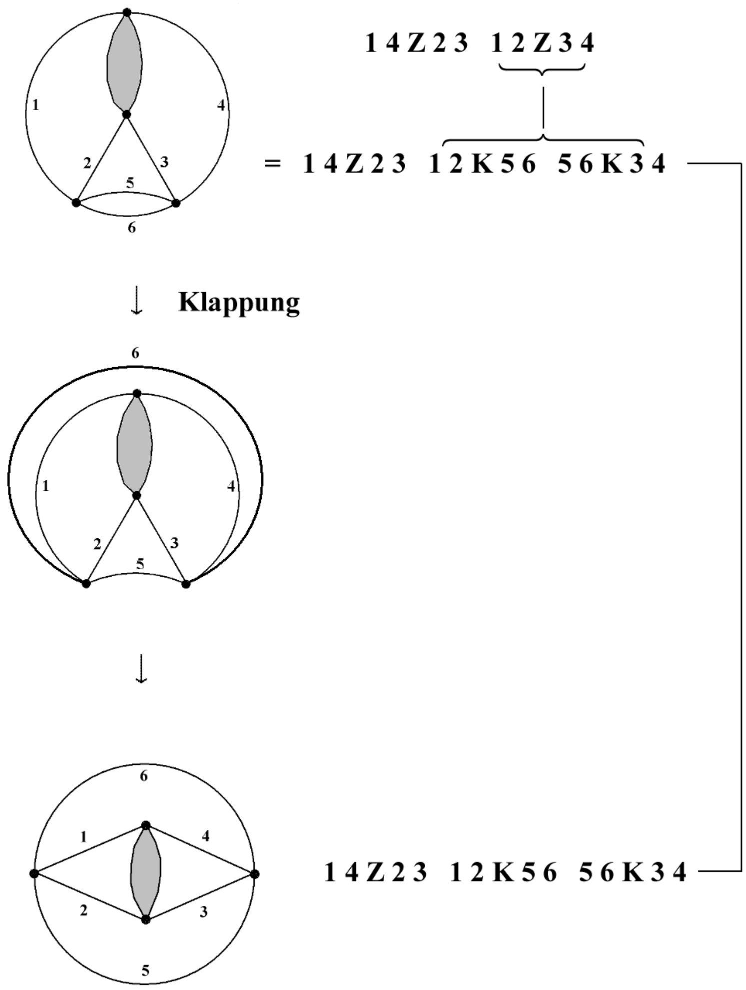


7_6

Knot graph reconstruction

1 2 Z 3 4	2 3 Z 5 6	1 5 Z Z 4 6




1 4 Z 2 3 1 2 Z 3 4
= 1 4 Z 2 3 1 2 K 5 6 5 6 K 3 4
Klappung
1 4 Z 2 3 1 2 K 5 6 5 6 K 3 4
1
2
3
4
5
6

7 Knot Graph Tables

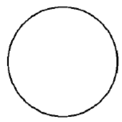
The following tables, beginning with the unknot, list systematically all extensions up to a crossing number of seven. All occurring variants of a knot graph or a link will be listed as a lookup table. In the next chapter the full context of the knot graph will be given and conclusions will be derived.

Legende

Kx.y.z	y.ter Knoten mit x Kreuzungen, z.te Variante
Vx.y.z	y.te Verschlingung mit x Kreuzungen, z.te Variante

Ursprung

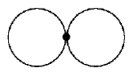
1 Nullknoten **K0.1**



Ursprung: K0.1

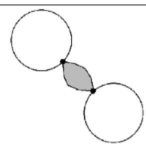
aus K0.1 durch Verdrehen entstanden bzw. durch Teilung

1 2 K 1 2 = 1 1 K 2 2 reduzierbarer Knoten K0.1 = **K1.1**



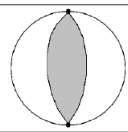
Ursprung: K1.1

1 1 Z 2 2 reduzierbarer Knoten = **K2.1**



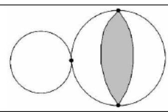
Ursprung: K1.1

1 2 Z 1 2 Verschlingung **V2.1**



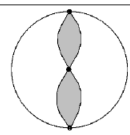
Ursprung: V2.1

1 1 K 2 3 2 4 Z 3 4 reduzierbare Verschlingung **V3.1**

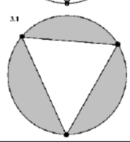


Ursprung: V2.1 und durch Teilung aus K1.1

1 2 Z Z 1 2 Knoten **K3.1.1**

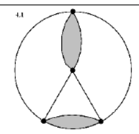


1 2 Z Z 1 2 Knoten **K3.1.2**



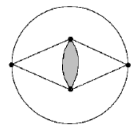
Ursprung: K3.1.1 und durch Teilung aus V2.1

1 3 Z 2 4 3 4 Z 1 2 Knoten **K4.1.1**



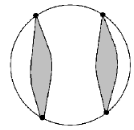
durch Törnen =

1 2 K 3 4 3 5 Z 4 6 5 6 K 1 2 Knoten **K4.1.2**



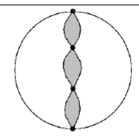
Ursprung: V3.1

1 2 Z 1 3 2 4 Z 3 4 Verschlingung **V4.0**

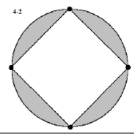


Ursprung: K3.1.1

1 2 Z Z Z 1 2 Verschlingung **V4.1.1**

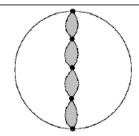


durch Klappung Verschlingung **V4.1.2**

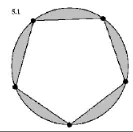


Ursprung: V4.1.1

1 2 Z Z Z Z 1 2 Knoten **K5.1.1**

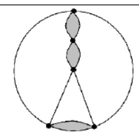


durch Klappung Knoten **K5.1.2**

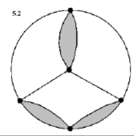


Ursprung: K4.1.1, K4.1.2, V4.1.1 und durch Teilung aus K3.1

1 2 Z Z 3 4 1 3 Z 2 4 Knoten **K5.2.1**

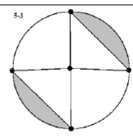


1 3 Z 2 4 3 4 Z Z 1 2 Knoten **K5.2.2**

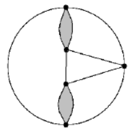


Ursprung: K4.1.1, K4.1.2 und durch Teilung aus K3.1

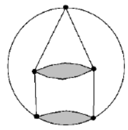
1 5 Z 2 6 5 6 K 3 4 1 3 Z 2 4 Verschlingung **V5.1.1**



durch Klappung Verschlingung **V5.1.2**
1 2 Z 3 6 2 3 K 4 5 4 6 Z 1 5

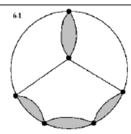


1 3 Z 2 4 3 5 Z 4 6 5 6 K 1 2 Verschlingung **V5.1.3**

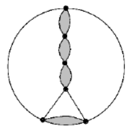


Ursprung: K5.2.1, K 5.1.1 und durch Teilung aus V4.1

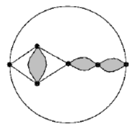
1 2 Z Z Z 3 4 1 3 Z 2 4 Knoten **K6.1.1**



durch Klappung
1 2 Z 3 4 1 3 Z Z Z 2 4 Knoten **K6.1.2**

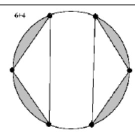


durch Törnen
1 2 K 3 4 3 5 Z 4 6 5 6 Z Z 1 2 Knoten **K6.1.3**

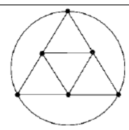


Ursprung: durch Teilung des reduzierbaren Knotens K2.1

1 2 Z Z 2 3 3 4 Z Z 1 4 nichtprimer Knoten **K6**

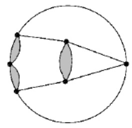


Ursprung: durch Teilung aus K3.1, V2.1
1 2 K 4 5 2 6 K 3 7 1 3 K 8 9 4 9 K 10 12 5 6 K 10 11 7 8 K 11 12
Verschlingung **V6.0**

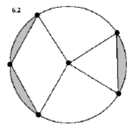


Ursprung: K5.2.1, V5.1.1, V5.1.2, V5.1.3 und durch Teilung aus V4.1

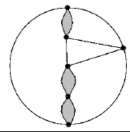
1 2 K 5 6 5 3 Z 6 4 1 3 Z Z 2 4 Knoten **K6.2.1**



1 5 Z Z 2 6 5 6 K 3 4 1 3 Z 2 4 Knoten **K6.2.2**

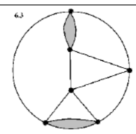


durch Klappung Knoten **K6.2.3**
1 2 K 5 6 5 3 Z 6 4 1 3 Z Z 2 4

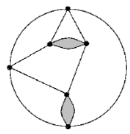


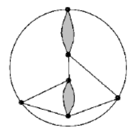
Ursprung: V5.1.1, V5.1.2, V5.1.3 und durch Teilung aus K4.1

1 7 Z 2 8 7 8 K 3 6 2 3 K 4 5 4 6 Z 1 5 Knoten **K6.3.1**

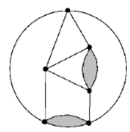


durch Klappung
1 5 K 7 8 7 2 Z 8 6 5 6 K 3 4 1 3 Z 2 4 Knoten **K6.3.2**



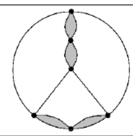


durch Klappung
1 2 K 5 6 5 3 Z 6 4 1 9 Z 3 10 9 10 K 2 4 Knoten **K6.3.3**



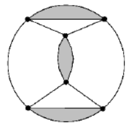
Ursprung: K5.2.1

1 2 Z Z 3 4 1 3 Z Z 2 4 Verschlingung **V6.1**



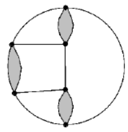
Ursprung: K5.2.1, K5.2.2, V5.1.1, V5.1.2, V5.1.3
und durch Teilung aus K4.1

1 5 Z 2 6 5 6 Z 3 4 1 3 Z 2 4 Verschlingung **V6.2.1**



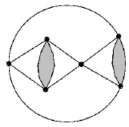
durch Klappung

1 2 Z 7 8 7 3 Z 8 4 1 3 Z 2 4 Verschlingung **V6.2.2**

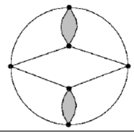


durch Törnen Verschlingung **V6.2.3**

1 2 K 5 6 5 7 Z 6 8 7 8 K 3 4 1 3 Z 2 4

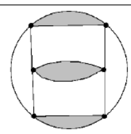


durch Klappung Verschlingung **V6.2.4**



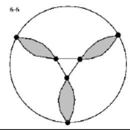
Ursprung: K5.2.2, V5.1.2, V5.1.3 und durch Teilung aus V4.1

1 2 Z 3 4 1 7 Z 3 8 7 8 Z 2 4 Verschlingung **V6.3.1**



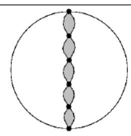
durch Klappung

1 2 Z 3 6 2 4 Z 3 5 4 6 Z 1 5 Verschlingung **V6.3.2**



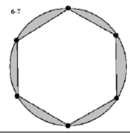
Ursprung: K 5.1.1

1 2 Z Z Z Z Z 1 2 Verschlingung **V6.4.1**



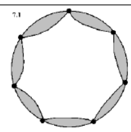
durch Klappung

1 3 Z 2 4 3 4 Z Z Z 1 2 Verschlingung **V6.4.2**



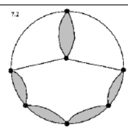
Ursprung: V6.4.1

1 2 Z Z Z Z Z Z 1 2 Knoten **K7.1**

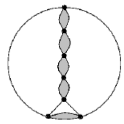


Ursprung: K6.1.1, K6.1.2, K6.1.3, V6.4.1 und durch Teilung aus K5.1

1 2 Z Z Z Z 3 4 1 3 Z 2 4 Knoten **K7.2.1**

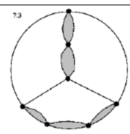


1 2 Z 3 4 1 4 Z Z Z Z 2 3 Knoten **K7.2.2**



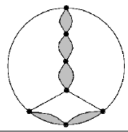
Ursprung: V6.1, K6.1.1, K6.1.2, K6.1.3, K6.2.2

1 2 Z Z Z 3 4 1 3 Z Z 2 4 Knoten **K7.3.1**



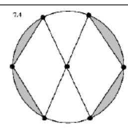
durch Klappung

1 2 Z Z 3 4 1 3 Z Z Z 2 4 Knoten **K7.3.2**



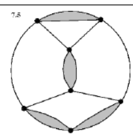
Ursprung: K6.2.2

1 2 Z Z 3 6 2 3 K 4 5 4 6 Z Z 1 5 Knoten **K7.4**

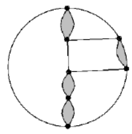


Ursprung: V6.1, V6.2.1, K6.2.2, V6.2.2 und durch Teilung aus K5.2

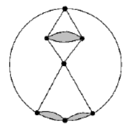
1 2 Z 3 6 2 3 Z 4 5 4 6 Z Z 1 5 Knoten **K7.5.1**



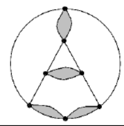
1 2 Z 5 6 3 5 Z Z 4 6 1 3 Z 2 4 Knoten **K7.5.2**



1 2 Z Z 3 4 1 3 K 9 10 9 11 Z 10 12 11 12 K 2 4 Knoten **K7.5.3**

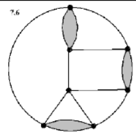


1 2 Z Z 3 4 1 9 Z 3 10 9 10 Z 2 4 Knoten **K7.5.4**

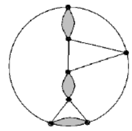


Ursprung: K6.2.2, K6.3.1, V6.2.1, V6.2.2
und durch Teilung aus K3.1, K5.2, V5.1

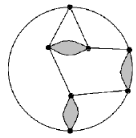
1 7 Z 2 8 7 8 K 3 6 2 3 Z 4 5 4 6 Z 1 5 Knoten **K7.6.1**



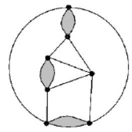
1 2 Z 3 6 2 3 Z 4 5 4 6 K 11 12 11 1 Z 12 5 Knoten **K7.6.2**



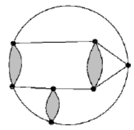
1 2 K 7 8 7 3 Z 8 6 2 3 Z 4 5 4 6 Z 1 5 Knoten **K7.6.3**



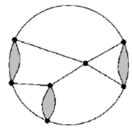
1 2 Z 5 6 3 5 K 9 10 9 4 Z 10 6 1 3 Z 2 4 Knoten **K7.6.4**



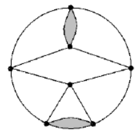
1 2 Z 5 6 3 5 Z 4 6 1 11 Z 3 12 11 12 K 2 4 Knoten **K7.6.5**



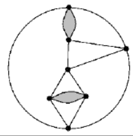
1 2 Z 9 10 9 3 Z 10 6 2 3 K 4 5 4 6 Z 1 5 Knoten **K7.6.6**



durch Törnen
1 2 Z 3 4 1 3 K 5 9 2 4 K 6 10 5 7 K 6 8 7 9 Z 8 10 Knoten **K7.6.7**

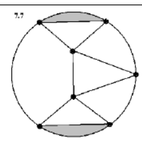


durch Törnen
1 2 K 7 8 7 9 Z 8 10 9 10 K 3 6 2 3 K 4 5 4 6 Z 1 5 Knoten **K7.6.8**

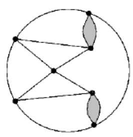


Ursprung: K6.3.1 und durch Teilung aus K3.1, V5.1

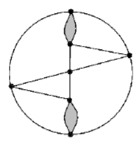
1 2 K 7 8 7 3 K 9 10 9 8 Z 10 6 2 3 K 4 5 4 6 Z 1 5 Knoten **K7.7.1**



1 2 K 7 8 7 3 Z 8 6 2 3 K 4 5 4 11 Z 6 12 11 12 K 1 5 Knoten **K7.7.2**

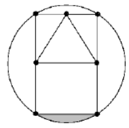


1 2 Z 8 10 1 6 K 4 5 6 8 K 7 9 2 3 K 9 10 3 4 Z 5 7 Knoten **K7.7.3**



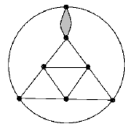
Ursprung: V6.0 und durch Teilung aus K3.1, K4.1, V4.1, V5.1
1 10 K 2 11 11 13 K 9 12 2 12 K 3 9 10 13 K 6 7 9 14 K 6 8 1 7 Z 3 8

Verschlingung **V7.0.1**



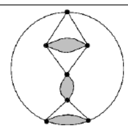
durch Klappung
1 10 K 2 11 11 13 K 9 12 2 12 K 3 9 10 13 K 6 7 9 14 K 6 8 1 7 Z 3 8

Verschlingung **V7.0.2**

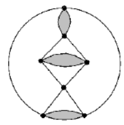


Ursprung: K6.1.1, K6.1.2, K6.1.3 und durch Teilung aus K3.1

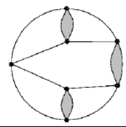
1 2 K 5 6 5 7 Z 6 8 7 8 Z 3 4 1 3 Z 2 4 Verschlingung **V7.1.1**



1 2 Z 7 8 7 9 Z 8 10 9 10 K 3 4 1 3 Z 2 4 Verschlingung **V7.1.2**

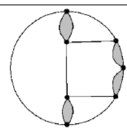


1 2 K 3 4 3 5 Z 4 6 5 6 Z 11 12 11 1 Z 12 2 Verschlingung **V7.1.3**

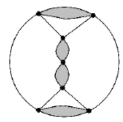


Ursprung: K6.1.1, K6.1.2, K6.1.3, V6.2.1, V6.2.2
und durch Teilung aus K5.2

1 5 Z 2 6 5 6 Z Z 3 4 1 3 Z 2 4 Verschlingung **V7.6.1**

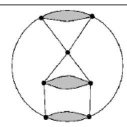


1 2 Z 3 6 2 3 Z Z 4 5 4 6 Z 1 5 Verschlingung **V7.6.2**

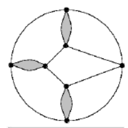


Ursprung: V6.2.1, V6.2.2 und durch Teilung aus K3.1, V5.1

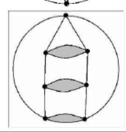
1 7 Z 2 8 7 8 K 5 6 3 5 Z 4 6 1 3 Z 2 4 Verschlingung **V7.2.1**



1 2 Z 3 4 1 7 Z 3 5 7 8 Z 5 6 2 4 K 6 8 Verschlingung **V7.2.2**

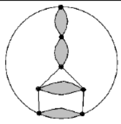


1 2 K 7 8 7 5 Z 8 6 3 5 Z 4 6 1 3 Z 2 4 Verschlingung **V7.2.3**

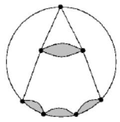


Ursprung: K6.1.1, K6.1.2, K6.1.3, K6.2.2 und durch Teilung aus K5.1

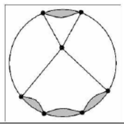
1 2 Z Z 9 10 9 3 Z 10 4 1 3 Z 2 4 Verschlingung **V7.3.1**



1 2 K 5 6 5 3 Z 6 4 1 3 Z Z Z 2 4 Verschlingung **V7.3.2**

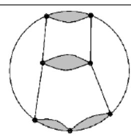


1 2 Z Z Z 3 6 2 3 K 4 5 4 6 Z 1 5 Verschlingung **V7.3.3**

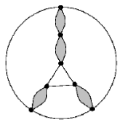


Ursprung: V6.3.2, V6.1, K6.2.2 und durch Teilung aus K5.1

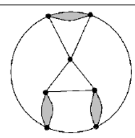
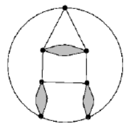
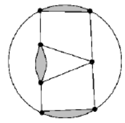
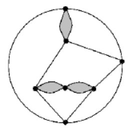
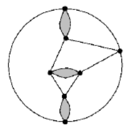
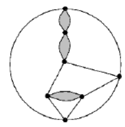
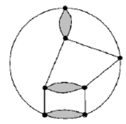
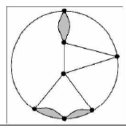
1 4 Z Z 2 3 2 5 Z 1 6 4 6 Z 3 5 Verschlingung **V7.4.1**



1 4 Z 2 3 2 5 Z Z 1 6 4 6 Z 3 5 Verschlingung **V7.4.2**



Ursprung: V6.3.2, K6.3.1, K6.2.2   und durch Teilung aus K5.2, V5.1

1 7 Z 4 8 7 8 K 2 3 2 5 Z 1 6 4 6 Z 3 5	Verschlingung **V7.5.1**	
1 4 K 7 8 7 2 Z 8 3 2 5 Z 1 6 4 6 Z 3 5	Verschlingung **V7.5.2**	
1 4 Z 2 3 2 5 Z 1 6 4 6 K 11 12 11 3 Z 12 5	Verschlingung **V7.5.3**	
1 2 K 7 8 7 3 Z Z 8 6 2 3 K 4 5 4 6 Z 1 5	Verschlingung **V7.5.4**	
1 2 Z 7 8 7 3 Z 8 6 2 3 K 4 5 4 6 Z 1 5	Verschlingung **V7.5.5**	
1 2 K 7 8 7 3 Z 8 6 2 3 K 4 5 4 6 Z Z 1 5	Verschlingung **V7.5.6**	
1 7 Z 2 8 7 3 Z 8 6 2 3 K 4 5 4 6 Z 1 5	Verschlingung **V7.5.7**	
1 2 Z Z 3 6 2 3 K 4 5 4 6 K 11 12 11 1 Z 12 5	Verschlingung **V7.5.8**	

For the translation of the german words please see the glossary.

8 Evaluation

8.1 Theorems

Different graphs of knots and links are related to each other by the present two types of extensions. Every graph, except the unknot, should have at least one predecessor and some successors.

Let $K = \{k_n \mid n \in \mathbb{N}\}$ be the set of all knot graphs and k_n a knot graph with n crossings. Each intersection can be extended "along" or "across". Thus, there are usually a large number of immediate extensions of a knot graph. Let, without restrictions, $E_r(k_n) = k_{n+1}$ be the r.th extension of the knot graph k_n. Then k_{n+1} is called a (immediate) **successor** of k_n , written $k_n \rhd k_{n+1}$.

Is the knot graph k_{n+x} generated by a succession of x consecutive extensions of k_n, so it is an (indirect) successor of k_n. The so generalized successor-relationship $k_m \blacktriangleright k_n$ (read: "from k_m follows k_n") is a relation $\blacktriangleright \subset K \times K$ on the set of the knot Graphs. (**Successor-Relation**). This relation is transitive, because if $k_i \blacktriangleright k_j$ and $k_j \blacktriangleright k_l$ then the combination of the sequences of extensions generates the successor relationship $k_i \blacktriangleright k_l$.

Because an extension always results in a knot graph with a higher number of crossings $k_n \blacktriangleright k_{n+x}$, so for k_n never a sequence of extensions exists that $k_{n+x} \blacktriangleright k_n$ would apply. The successor relation is therefore asymmetrically.

Thus:

Theorem 1: The successor relation is an order relation. ■

On the set of knot Graphs, therefore an order structure can be defined (See Chapter 8.3).

The reductions to distinguish knots from links form a different kind of relation. We can classify the set of the knot graphs into disjoint subsets, if the last step of the reduction of each knot into the unknot is excluded.

If we introduce the identical reduction (which leaves everything unchanged) and define the inverted reductions as structure-preserving extensions, then, sequences of reductions are reflexive, transitive, and symmetric and it follows:

Theorem 2: Sequences of reductions are equivalence relations between knot graphs of different number of crossing. ■

Still there are the previously defined equivalence operations for transforming a knot graph into another with same number of crossings. If the operation which leaves everything unchanged is introduced as identical equivalence operation, then sequences of this equivalence operations are reflexive, transitive, and symmetrical. We have therefore:

Theorem 3: Sequences of equivalence operations are equivalence relations between knot graphs of the same intersection number. ■

Theorem 4: With the exception of the unknot is the number of strands in a knot graph twice the number of crossings, as an intersection is formed by two strands. ■

Probably, these theorems allow more statements about the connection of the relations, infima, maximum and minimum elements, etc. They may contribute to the classification and ultimately to the calculation of knot graphs or finally knots.

8.2 Knot Graph Genealogy

Here follows the list of derivations of the knot graphs, their family relationships, so to speak:

K0.1 is (reducibly) splitted into K1.1

K1.1 splits into K3.1 and expands to K2.1 (reducible), V2.1

V2.1 expands to K3.1

K3.1 is splitted into K5.1, K5.2, K7.6, K7.7, V6.0, V7.0, V7.1, V7.2 and expands to K4.1, V4.1

K4.1 is splitted into K6.3, V6.2, V7.0 and expands to K5.2, V5.1

V4.1 is splitted into K6.1, K6.2, K8.18, K9.40, V6.3, V7.0 and expands to K5.1, K5.2

K5.1 is splitted into K7.2, V7.3, V7.4 and expands to K6.1, V6.1

K5.2 is splitted into K7.5, K7.6, V7.5, V7.6 and expands to K6.1, K6.2, V6.2, V6.3

V5.1 is splitted into K7.6, K7.7, V7.0, V7.2, V7.5 and expands to K6.2, K6.3, V6.2, V6.3, V6.4

K6.1 expands to K7.2, K7.3, V7.1, V7.3, V7.6

K6.2 expands to K7.3, K7.4, K7.5, K7.6, V7.3, V7.4, V7.5

K6.3 expands to K7.6, K7.7, V7.5

V6.0 expands to V7.0

V6.1 expands to K7.3, K7.5, V7.4

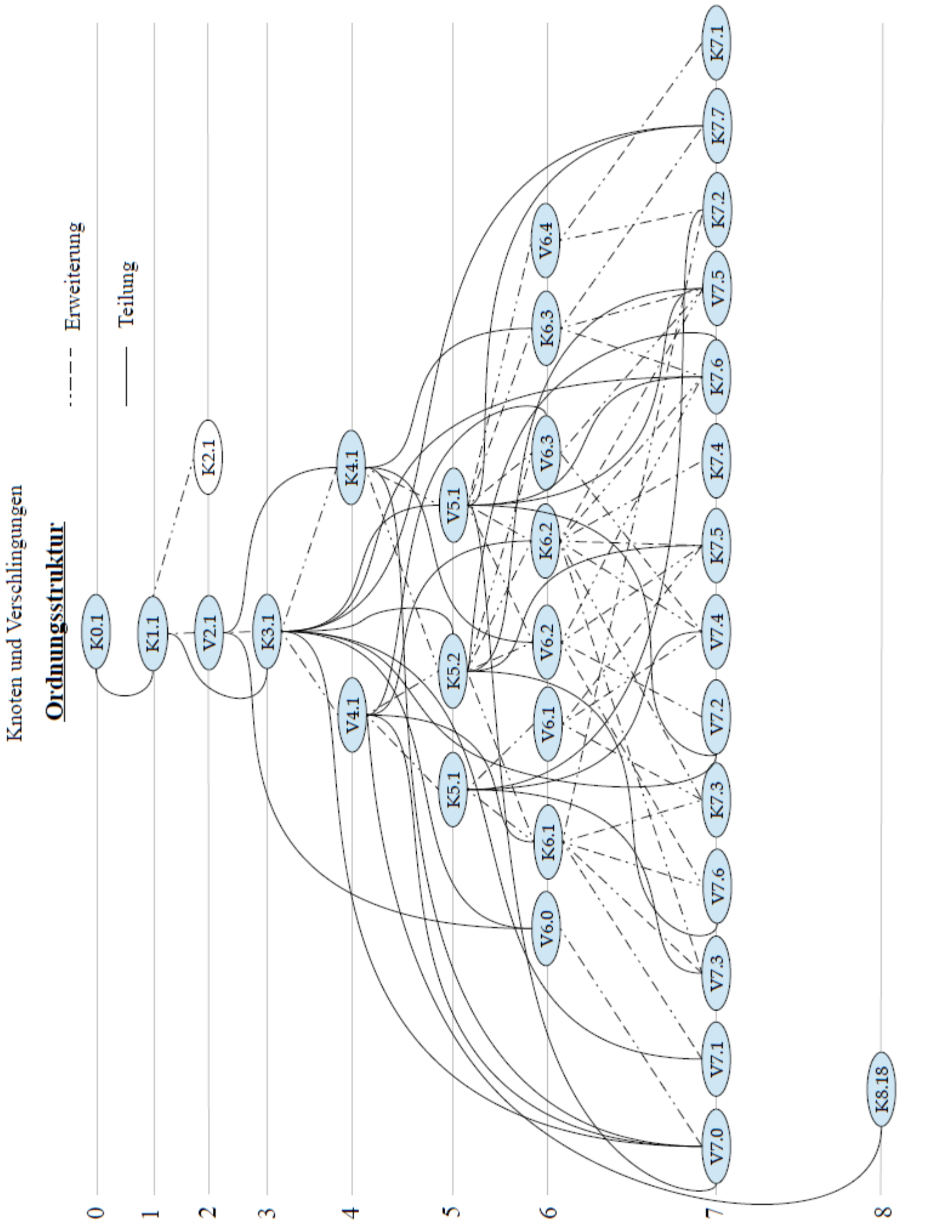
V6.2 expands to K7.5, K7.6, V7.2, V7.6

V6.3 expands to V7.4, V7.5

V6.4 expands to K7.1, K7.2

8.3 Order Structure

The order relation, defined accordingly to Theorem 1), causes the following order structure on the set of knot graphs:



8.4 Equivalence Classes of the Reduction

The equivalence relation according to theorem 2) defined between knot graph classifies the set of the knot graphs of different numbers of crossings into equivalence classes.

Ultimately, all prime knots are reduced to the unknot.

However, there exists no reduction of the figure eight knot with four intersections into the overhand knot with three crossings, but only the direct reduction into the unknot.

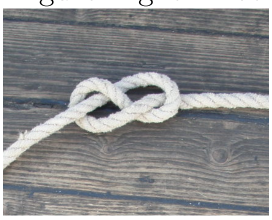
If we omit the last reduction into the unknot, we can reduce all knot graphs either into the figure eight knot or into the overhand knot and, thus, into only two equivalence classes.

One class is represented by the overhand knot:

Overhand Knot

and the other class is represented by the figure eight knot:

Figure Eight Knot



Remark: Boatswain Stempel of the Sailor School Hamburg Finkenwerder taught us wimps, that all utility knots had been developed out of the overhand knot or the figure eight knot.

Boatswain Stempel 1977

He was obviously right. So far, there is at least no known counterexample (see Conjecture 3).

8.5 Canonical Map: Equivalence Class Membership of the Knot Graphs

K3.1 is the overhand knot

K4.1 is the figure eight knot

K5.1 reduces to the overhand knot

K5.2 reduces to the overhand knot

K6.1 reduces to the figure eight knot

K6.2 reduces to the figure eight knot

K6.3 reduces to the overhand knot

K7.1 reduces to the overhand knot

K7.2 reduces to the overhand knot

K7.3 reduces to the overhand knot

K7.4 reduces to the overhand knot

K7.5 reduces to the overhand knot

K7.6 reduces to the overhand knot

K7.7 reduces to the figure eight knot

K8.1 reduces to the figure eight knot

K8.2 reduces to the figure eight knot

K8.3 reduces to the figure eight knot

K8.4 reduces to the figure eight knot

K8.5 reduces to the figure eight knot

K8.6 reduces to the figure eight knot

K8.7 reduces to the overhand knot

K8.8 reduces to the overhand knot

K8.9 reduces to the figure eight knot

K8.10 reduces to the overhand knot

K8.11 reduces to the figure eight knot

K8.12 reduces to the figure eight knot

K8.13 reduces to the overhand knot

K8.14 reduces to the figure eight knot

K8.15 reduces to the overhand knot

K8.16 reduces to the overhand knot

K8.17 reduces to the figure eight knot

K8.18 reduces the overhand knot

K8.19 reduces to the overhand knot

K8.20 reduces to the overhand knot

K8.21 reduces to the overhand knot

K9.1 reduces to the overhand knot

K9.2 reduces to the overhand knot

K9.3 reduces to the overhand knot

K9.4 reduces to the overhand knot

K9.5 reduces to the overhand knot

K9.6 reduces to the overhand knot

K9.7 reduces to the overhand knot

K9.8 reduces to the overhand knot

K9.9 reduces to the overhand knot

K9.10 reduces to the overhand knot

K9.11 reduces to the overhand knot

K9.12 reduces to the overhand knot

K9.13 reduces to the overhand knot

K9.14 reduces to the figure eight knot

K9.15 reduces to the overhand knot

K9.16 reduces to the overhand knot

K9.17 reduces to the figure eight knot

K9.18 reduces the overhand knot

K9.19 reduces to the figure eight knot

K9.20 reduces to the overhand knot

K9.21 reduces to the overhand knot

K9.22 reduces to the figure eight knot

K9.23 reduces to the overhand knot

K9.24 reduces to the overhand knot

K9.25 reduces to the overhand knot

K9.26 reduces to the figure eight knot

K9.27 reduces to the figure eight knot

K9.28 reduces to the overhand knot

K9.29 reduces to the overhand knot

K9.30 reduces to the figure eight knot

K9.31 reduces to the overhand knot

K9.32 reduces to the figure eight knot

K9.33 reduces to the figure eight knot

K9.34 reduces to the figure eight knot

K9.35 reduces to the overhand knot

K9.36 reduces to the figure eight knot

K9.37 reduces to the figure eight knot

K9.38 reduces to the overhand knot

K9.39 reduces to the figure eight knot

K9.40 reduces to the figure eight knot

K9.40 reduces to the figure eight knot

K9.41 reduces to the figure eight knot

K9.42 reduces to the figure eight knot

K9.44 reduces to the overhand knot

K9.45 reduces to the figure eight knot

K9.46 reduces to the overhand knot

K9.47 reduces to the figure eight knot

K9.48 reduces to the figure eight knot

K9.49 reduces to the figure eight knot

K10.1, Perko 1, reduces to the overhand knot

K10.2, Perko 2, reduces to the figure eight knot

8.6 Degree of the Link Graphs

Links can be classified after reduction according to their degree.

V2.1 is of degree 2

V3.1 is of degree 2

V4.0 is of degree 3

V4.1 is of degree 2

V5.1 is of degree 2

V6.0 is of degree 3

V6.1 is of degree 2

V6.2 is of degree 2

V6.3 is of degree 3

V6.4 is of degree 2

V7.0 is of degree 2

V7.1 is of degree 2

V7.2 is of degree 3

V7.3 is of degree 2

V7.4 is of degree 2

V7.5 is of degree 2

V7.6 is of degree 2

The equivalence relation within the knot graphs of equal crossing number, defined according to theorem 3) ensures the uniqueness of the knot graphs.

8.7 Conjectures

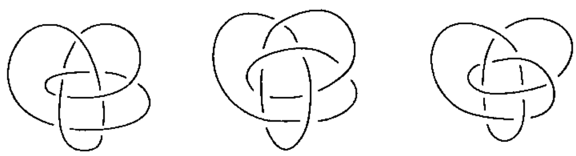
Here are some obvious conjectures, but their proofs are not yet sought.

Conjecture 1: The order structure of the knot graphs under the successor relation is a partial order or lattice. □

Conjecture 2: A tangle is always non-alternating. □

Corollary: But not every non-alternating knot diagram is a tangle. ■

Counter-examples: The knots K8.19, K8.20, K8.21, etc.:



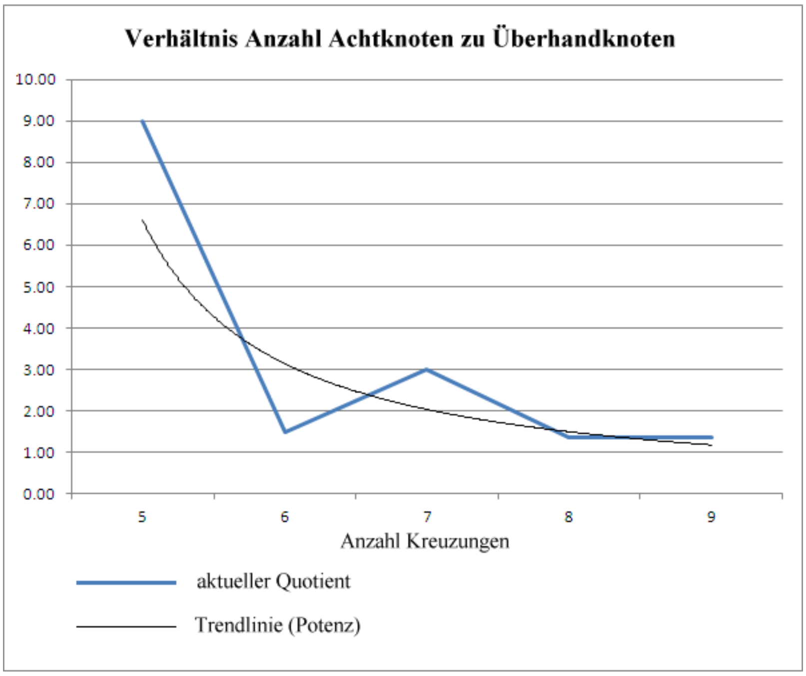
8_{19} 8_{20} 8_{21}

Conjecture 3: All knot graphs of real knots can be reduced either to the overhand knot or the figure eight knot. □

Conjecture 4: A knot graph with overlapping strands and an odd number of crossings is planar, if the sum of the number of overlappings, intersections and bi-angles is odd, otherwise it is non-planar. □

Conjecture 5: Most knots belong to the class of overhand knot. The ratio of overhand knot reductions to figure eight knot reductions is so far $\frac{47}{35} = 1.34$. □

Conjecture 6: For reductions to the figure eight knot compared to the overhand knot the ratio converges against a fixed number in the range of 1. □



8.8 Further Questions

Further questions can be asked, e.g.:

- How can the ambiguities of the knot graphs be solved?
- Is the re-drawn knot graph of a tangle always only another knot, or can it also be a tangle? Here another field for investigations is opened.
- Which knot graphs are elements of the equivalence classes of the reduction. What conclusions follow for the classification of the knots?
- Are the so far given equivalence operations sufficient to identify knot graphs with the same number of crossings? Is this also true for the infinite number of knot graphs or will it be always neccessary to formulate new equivalence transformations to detect increasingly complex knot graphs as equal?
- Could it happen that reducable knots arise while expanding knot graphs?

- What is the maximum degree links have with given number of crossings?
- How many bi-angles at maximum can a knot graph have at a given number of crossings?
- As the overhand knot and the figure eight knot reduce directly to the unknot, they could be described as **fundamental knots**. The question arises whether there are more fundamental knots which reduce directly to the unknot, without intermediate steps. Under the reviewed knots with crossing numbers up to 10 there is no more fundamental knot - remain infinitely more to check.
- Can the order and equivalence relations be found in the known knot invariants (such as the Jones - or Conway polynomials)?

8.9 Development of the Methods

The expansion operations of an intersection in a bi-angle, "across" or "along", were not sufficient to generate all knot graphs. The translation into conventional knot diagrams with equal crossing number was managed easily so far, but it lacked at least one rule because there were knots such as the turks head knot that could not be created.

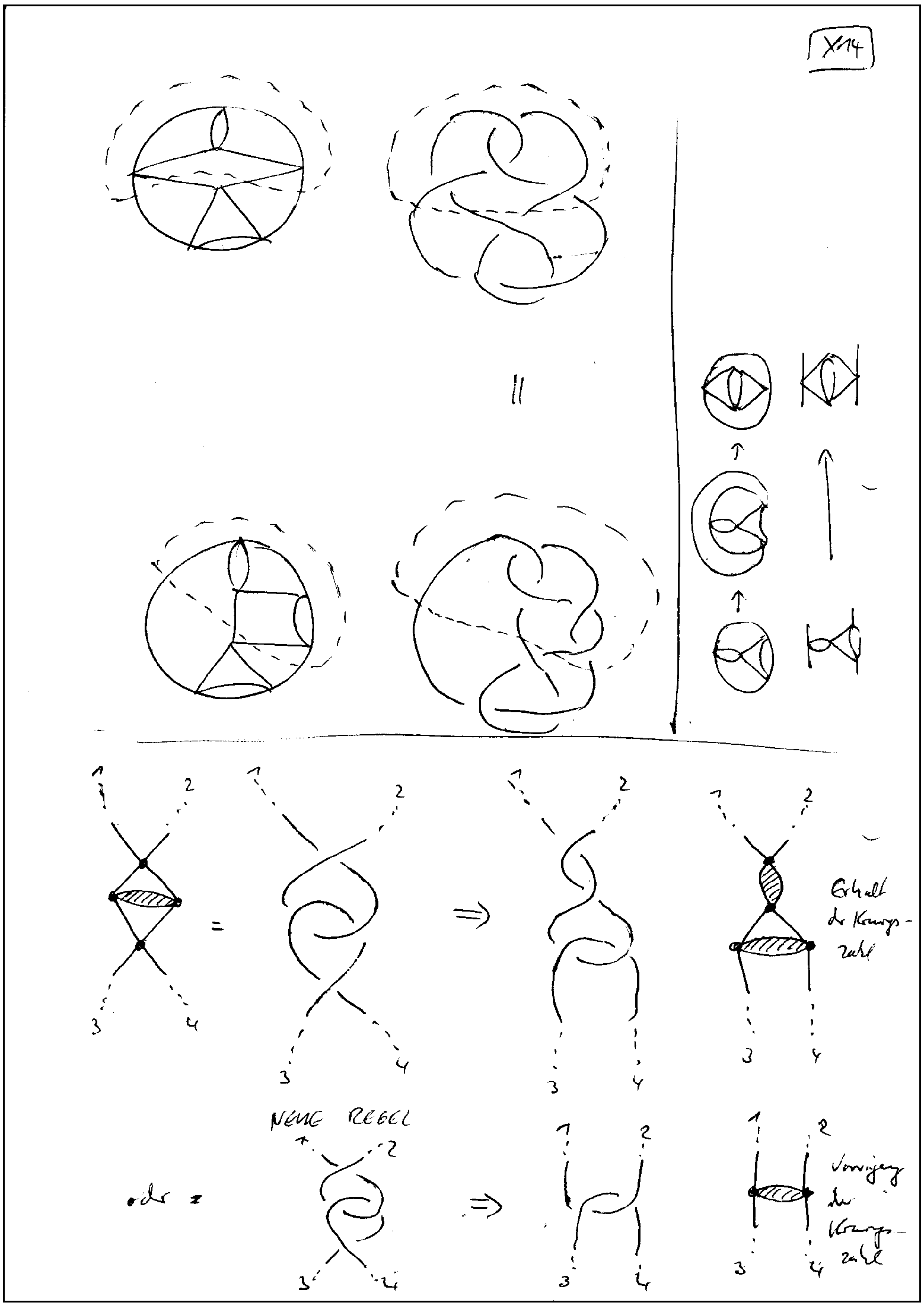
Sometimes the wording of a rule succeeded only by recourse to the "semantics" of the knot graph, therefore, the corresponding appropriate knot diagrams, and playing around with the appropriate knot. I used occasionally real ropes to see more clearly.

The splitting rule was introduced to be able to create the turks head knot and similar knots. Also the reduction after expansion, to be able to reduce bi-angle free knot graphs.

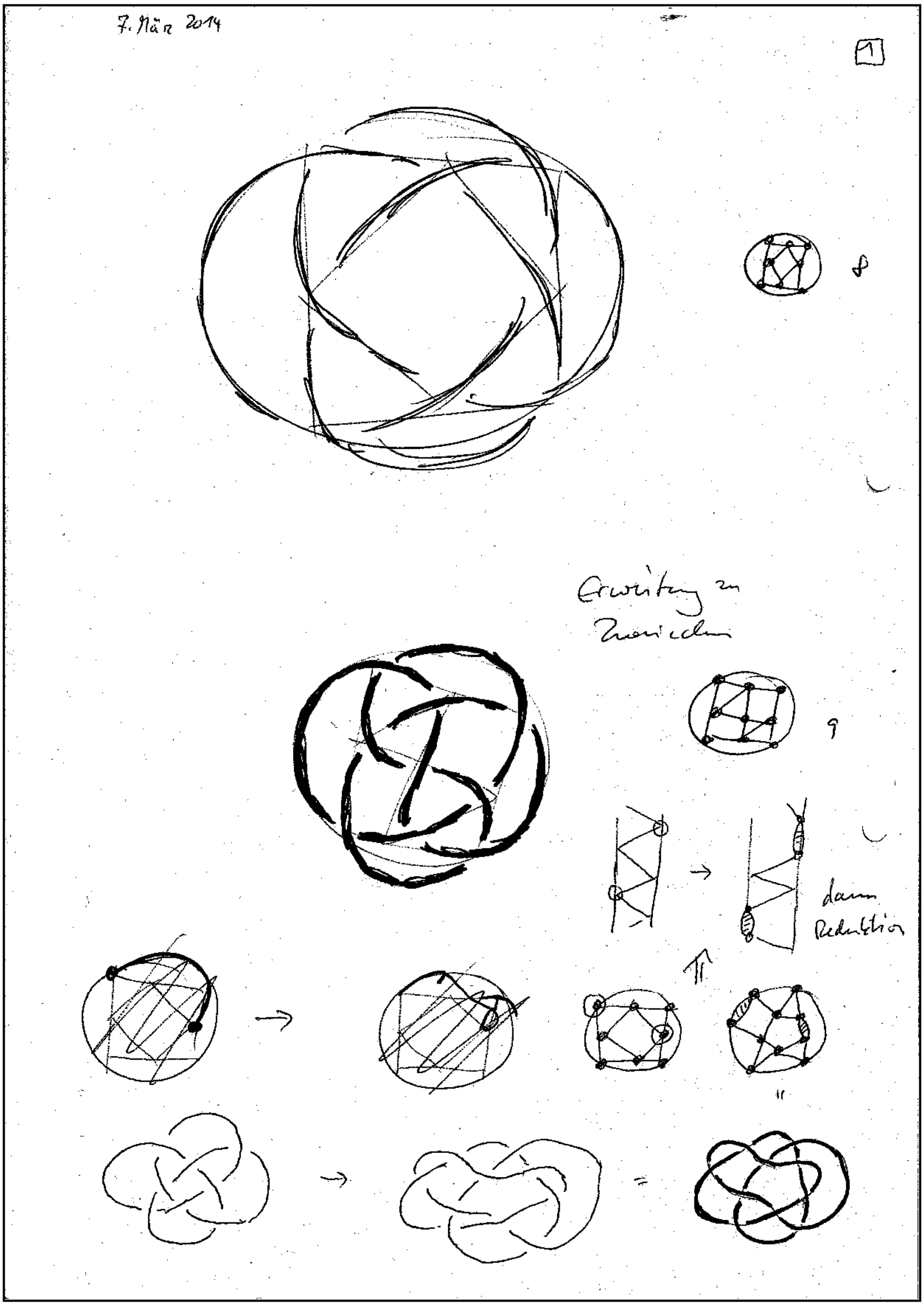
Here are some examples of original worksheets that inspired the presented Ideas:

For example, the rule of for twisting had to be introduced to identify the knot graph K7.6.7 as equivalent to other knots of the class K7.6:

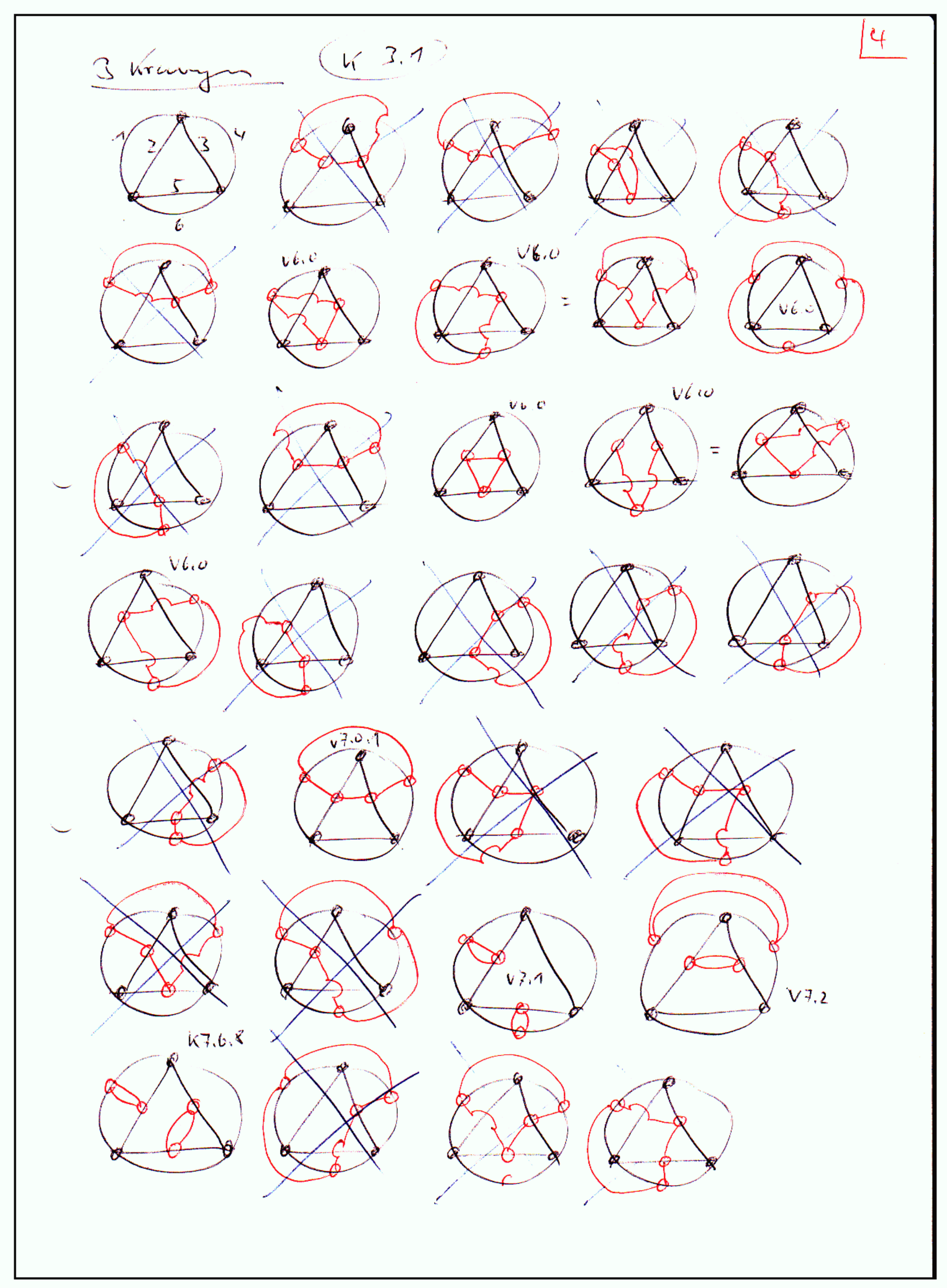
Original drawing to the twisting rule



Original drawing to the "reduction by expansion" rule



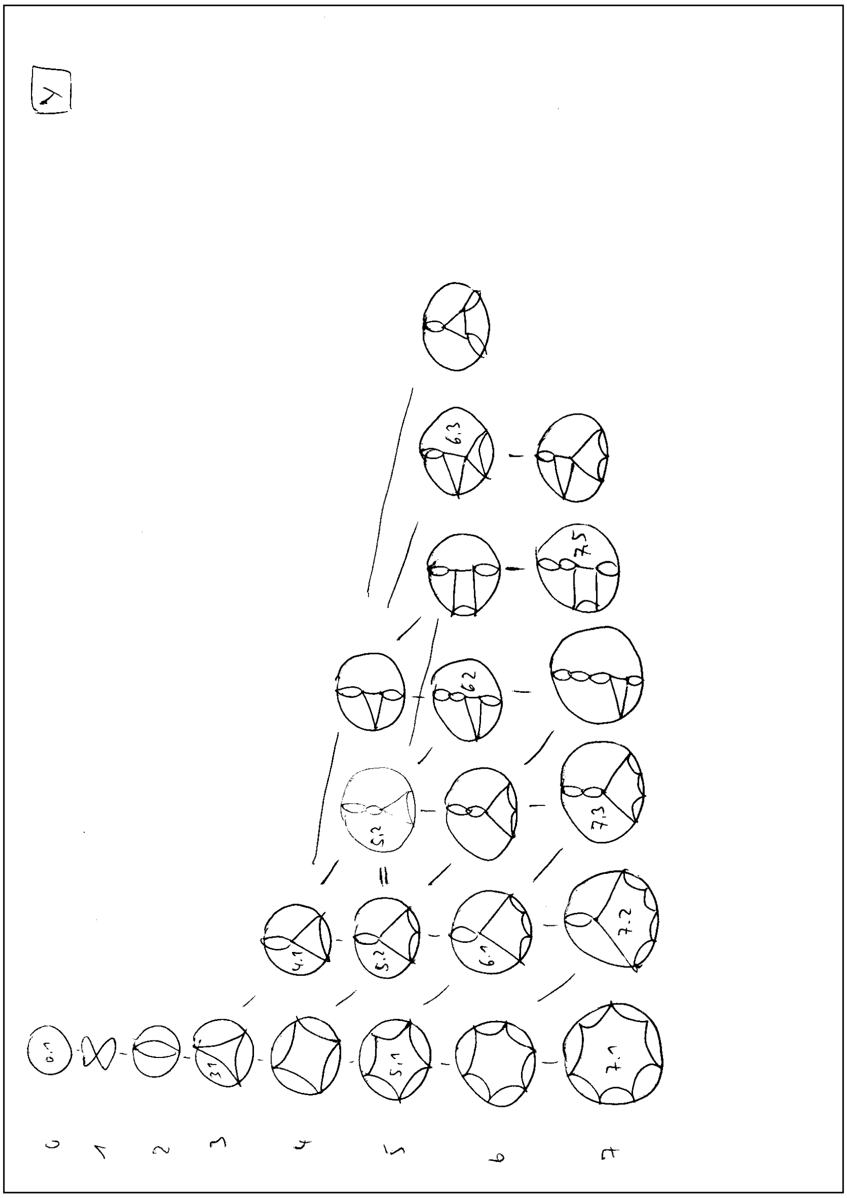
Example of derivations: Splitting the overhand knot



The beginning was the presumption, that the knot graphs could be sorted into an order-structure graph, that looks about as shown in the following image.

The hope was that alone from the position in the order-structure table it could be determined whether it is a knot or a link. This turned eventually out not to be true:

Original drawing for the order-structure graph



8.10 Problems

Beside the tasks further mentioned in the questions there are still open issues which should be resolved. In addition to the question of whether the material collection is correct as it is now, the question is, whether the definitions of relations and the resulting theorems are mathematically correct.

8.11 Correctness of Theorems

The definitions, relations, and theorems have to be reviewed for correctness and accuracy of the proofs. This should be done by mathematicians.

8.12 Accuracy

The check the completeness and accuracy of the knot graphs which represent the prime knots one can easily use the knot diagrams published in the literature. It is more difficult in the case of the links, for there are no comparable tables published. Here both, the question of completeness as well as the problem of not yet discovered identities still remains. I tried all efforts to ensure the accuracy, but (still) can not guarantee it.

9 Exercises

1. Create the knot graph described by
   1 2 K 7 8 7 3 K 9 10 9 8 Z 10 6 2 3 K 4 5 4 6 Z 1 5
2. Show the equivalence of the knot graphs K7.6.1 and K7.6.7
3. Reduce the knot graph K7.5 2 to the unknot. Does the reduction lead via the figure eight knot or via the overhand knot?
4. Create a knot graph with seven crossings from the knot graph K6.2.2 by extending one single crossing.
5. Is this the graph of a knot or of a link?
6. Reduce the knot graph V6.0.

10 Sources

References

[1] Charles Livingston, Knotentheorie für Einsteiger
Vieweg Verlag, Braunschweig/Wiesbaden 1995

[2] Alexei Sossinsky, Mathematik der Knoten
Rowohlt Verlag, Reinbek bei Hamburg 2000

[3] Peter Tittmann, Graphentheorie
Carl Hanser Verlag, München 2011

[4] Clifford W. Ashley, The Ashley Book of Knots
Faber and Faber, London 1977