Calculation of the Speed of Sound

Wolf Scheuermann, November 1998.

Hook's law is applicable for elastic material with linear response to an external force, like springs:
Forces are proportional to the change of length.
A gas can be described as a spring (- and used: e.g. pneumatic cushioning in trucks).

Imagine a pipe with **cross section A** filled with a gas of **density** ρ_0. A volume element of **length dx** is set under a **force dF** in longitudinal direction x of the pipe. The length of the volume element changes with the force by the amount dξ. A **pressure dp** acting in opposite direction of the force will be felt.

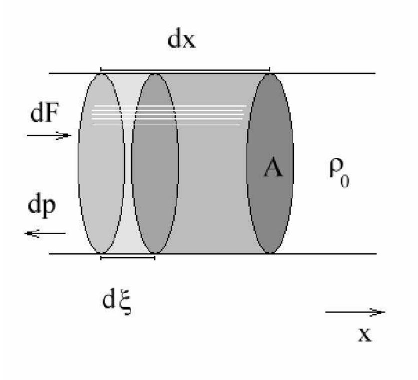

Figure 1. A Volume Element of a Gas under Pressure.

Eq. 1)

$$\boxed{\mathrm{d}p = -\frac{\mathrm{d}F}{A}}$$

According to Newton:

$$\textit{Force} = \textit{Mass} \cdot \textit{Acceleration}$$

$$F = M \cdot a \qquad \textit{with} \quad a = \frac{\mathrm{d}^2\xi}{\mathrm{d}t^2}$$

$$\rightarrow \quad \mathrm{d}F = -A \cdot \mathrm{d}p = \mathrm{d}M \cdot a = \rho_0\, \mathrm{d}V \cdot a = \rho_0 A\, \mathrm{d}x \cdot \frac{\mathrm{d}^2\xi}{\mathrm{d}t^2}$$

$$\rightarrow \quad -A\, \mathrm{d}p = \rho_0 A\, \mathrm{d}x \cdot \frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} \quad \rightarrow$$

Eq. 2)

$$\boxed{\frac{\mathrm{d}p}{\mathrm{d}x} = -\rho_0 \cdot \frac{\mathrm{d}^2\xi}{\mathrm{d}t^2}}$$

On the other hand the pressure can be described using the change of length of the volume element:

$$\mathrm{d}p \sim -\frac{\mathrm{d}\xi}{\mathrm{d}x}$$

The proportional action factor is the **modulus of elasticity E**. For gases, the **modulus of compression** [same unit like pressure] is used.

We get

Eq. 3) $$\boxed{\mathrm{d}p = -E \cdot \frac{\mathrm{d}\xi}{\mathrm{d}x}}$$

Derivation with respect to x delivers the result:

Eq. 4) $$\boxed{\frac{\mathrm{d}p}{\mathrm{d}x} = -E \cdot \frac{\mathrm{d}^2\xi}{\mathrm{d}x^2}}$$

$$\rightarrow \qquad -\rho_0 \cdot \frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} = -E \cdot \frac{\mathrm{d}^2\xi}{\mathrm{d}x^2}$$

This is indeed a special form of the wave equation

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} + c^2 \cdot \frac{\mathrm{d}^2\xi}{\mathrm{d}x^2} = 0$$

where **c** is the **wave propagation speed**.

With Eq. 2, 4

$$\rightarrow \qquad \frac{\mathrm{d}x^2}{\mathrm{d}t^2} = \frac{E}{\rho_0} \cdot \frac{\mathrm{d}^2\xi}{\mathrm{d}^2\xi}$$

$$\begin{matrix} \downarrow & & & \downarrow \\ c^2 & = & \dfrac{E}{\rho_0} \cdot & 1 \end{matrix} \qquad \rightarrow \qquad c^2 = \frac{E}{\rho_0} \qquad \rightarrow$$

Speed of Sound

Eq. 5) $$\boxed{c = \sqrt{\frac{E}{\rho_0}}}$$

Because sound can be described as a pressure wave as well as a density wave, the change of the density of the material is also determined by the change of length:

Eq. 6) $$\boxed{\mathrm{d}\rho = -\rho_0 \cdot \frac{\mathrm{d}\xi}{\mathrm{d}x}}$$

Expansion with c²

$$\rightarrow \qquad c^2 \cdot \mathrm{d}\rho = -\rho_0 \cdot c^2 \cdot \frac{\mathrm{d}\xi}{\mathrm{d}x}$$

With Eq. 5 and Eq.3 $\rightarrow$ $c^2 \cdot \mathrm{d}\rho = -\rho_0 \cdot \frac{E}{\rho_0} \cdot \frac{\mathrm{d}\xi}{\mathrm{d}x} = -E \cdot \frac{\mathrm{d}\xi}{\mathrm{d}x} = \mathrm{d}p$ $\rightarrow$

General Definition of the Speed of Sound in Elastic Material:

Eq. 7) $$\boxed{c^2 = \frac{\mathrm{d}p}{\mathrm{d}\rho}}$$

Compression of gases (and sound is a phenomenon of compression) can be described as an **adiabatic process**, a process where no heat exchange occurs.

For adiabatic processes the following equation is valid:

Eq. 8) $$\boxed{p = C \cdot \rho^{\gamma}}$$

γ: **adiabatic exponent**. For air (or diatomic gases: N_2+O_2): $\boldsymbol{\gamma = 1.4}$

Derivation of Eq. 8 with respect to ρ and use of Eq. 7 and 5

$$\rightarrow \quad \frac{\mathrm{d}p}{\mathrm{d}\rho} = \gamma \cdot C \cdot \rho^{\gamma - 1} = c^2 = \frac{E}{\rho} \quad \rightarrow \quad E = \gamma \cdot C \cdot \rho^{\gamma} \quad \rightarrow$$

with Eq. 8

Eq. 9) $$\boxed{E = \gamma \cdot p}$$

With the **equation of ideal gases**

Eq. 10) $$\boxed{p = \rho \cdot R \cdot T}$$

where R is the gas constant: $R = 287 \frac{J}{kg \cdot K}$ for air,

and Eq. 9 follows

$$E = \gamma \cdot \rho \cdot R \cdot T$$

with Eq. 5

$$\rightarrow \quad c = \sqrt{\frac{\gamma \cdot \rho \cdot R \cdot T}{\rho}} \quad \rightarrow$$

Speed of Sound in Air

Eq.11) $$\boxed{c = \sqrt{\gamma \cdot R \cdot T}}$$

$$\rightarrow \qquad c = \sqrt{1.4 \cdot 287 \frac{J}{kg \cdot K} \cdot T} = \sqrt{1.4 \cdot 287 \frac{Nm}{kg \cdot K} \cdot T} = \sqrt{1.4 \cdot 287 \frac{kg \cdot \frac{m}{s^2} \cdot m}{kg \cdot K} \cdot T} = 20.045 \frac{m}{s} \cdot \frac{1}{\sqrt{K}} \cdot \sqrt{T}$$

This result may be used in a **Rule of Thumb** for calculation of the speed of sound in air:

Eq. 12) $$\boxed{c \approx 20 \cdot \sqrt{T}}$$

T: OAT in Kelvin

c in $\frac{m}{s}$.

Note: The speed of sound in gases only depends on the temperature of the gas, not on the density.
Example:

$$T_o = 288.15\,K \qquad \rightarrow \qquad c = 340.26 \frac{m}{s}$$

$$\rightarrow \qquad c = 661.42\ kt$$

The rule of thumb delivers:

$$c \approx 20 \cdot \sqrt{T_0} = 20 \cdot \sqrt{288} = 339 \quad \frac{m}{s}$$

a quite good result.

This is valid for a great range of temperatures.

Literature

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